points might, for instance, be simply end points. These contentions are directed against the atomists. The arguments are wanting in explicitness and precision. What we said of Aquinas's commentary on Zeno applies also to Duns Scotus.[1] He gives detailed elaborations of Aristotle without offering new explanations of Zeno's puzzles. In place of Achilles and the tortoise he introduces the more familiar travelers, the horse and the ant. His commentaries are annotated by the Franciscan theologian Franciscus de Pitigianis of Arezzo in Italy, who wrote the latter part of the sixteenth century. This annotator expresses himself in favor of the admission of the actual infinity to explain the "Dichotomy" and the "Achilles," but fails to adequately elaborate the subject. Scholastic ideas on infinity and the continuum find expression in the writings of Bradwardine, the English *doctor profundus*. He says that five explanations have been given of the nature of the continuum.[2]

GROUPS OF SUBTRACTION AND DIVISION WITH RESPECT TO A MODULUS.

By G. A. MILLER, University of Illinois.

Certain kinds of groups of subtraction and division were explained by the present writer in two articles entitled: "Groups of the fundamental operations of arithmetic" and "Groups of subtraction and division." These articles were published respectively in the *Annals of Mathematics*, volume 6 (1905), page 41; and in the *Quarterly Journal of Mathematics*, volume 37 (1906), page 80. The present article is devoted to more elementary considerations, and has for its main object to exhibit interesting elementary relations between certain groups of subtraction and division, and the corresponding groups of addition and multiplication.

It is well known, and also evident, that the first $m-1$ natural numbers together with zero constitute the cyclic group of order m with respect to addition when the sums are replaced by their least positive residues, or by zero, modulo m. That is, if in the series of numbers

$$0,\ 1,\ 2,\ \cdots,\ m-1$$

each number is replaced by itself increased by α, mod m, where $0 \lesseqgtr \alpha \lesseqgtr m-1$, there results a certain substitution on these m numbers, and the totality of the distinct substitutions which can be constructed in this manner constitutes the cyclic group of order m. The order of the substitution corresponding to α is

[1] Duns Scoti, *Opera Omnia*, T. II: Joannis Duns Scoti Doctoris Subtilis, ordinis minorum, in VIII libros Physicorum Aristotelis Quæstiones, cum annotationibus R. P. F. Francisci Pitigiani arretini, etc. Lvgdvni, MDCXXXIX, Quaestio X, pp. 390–393.

[2] See Maximilian Curtze on the "Tractatus de continuo Bradwardini" in *Zeitschrift f. math. u. Phys.*, XIII Jahrg., Suppl., 1868, Leipzig, p. 88.

evidently the quotient obtained by dividing m by the highest common factor of α and m. In particular, when $\alpha = 0$ this substitution reduces to the identity.

If each of the numbers of the given series is replaced by itself decreased by α, mod m, there results again a substitution, and the totality of the distinct substitutions which can be obtained in this manner constitutes again the cyclic group of order m. In fact, the substitution which corresponds to α when α is subtracted from each of the given numbers is the inverse of the substitution which corresponds to α when it is added to each of these numbers, since the successive performance of these two operations leaves each of the given numbers unchanged. As an automorphism of any abelian group can be established by letting each operator of this group correspond to its inverse, it results that a simple isomorphism between the given group of addition and the given group of subtraction can be established in such a way that the operations which result from the same number in the processes of addition and subtraction correspond in this simple isomorphism.

The simple isomorphism which has been considered exhibits very clearly that the operations of subtracting successively each of the numbers of the given series from every number of this series have the same relative properties as those of adding these same numbers to every number of this series. The fact that each operator corresponds to its inverse in the given simple isomorphism may be regarded as an extension of the concept that addition and subtraction are inverse operations. In fact, not only are these operators inverses but the two groups which they constitute under the given conditions are such that every operator of the one corresponds to the inverse of the other.

When the $\phi(m)$ natural numbers which do not exceed m and are prime to m are combined by multiplication, mod m, they form one of the most important classes of abelian groups. For any particular value of m, the corresponding group of order $\varphi(m)$ can be obtained by replacing these $\phi(m)$ numbers by the set obtained by multiplying each of them by k, provided k is prime to m and mod m is replaced by mod km. In fact, it has been observed that a necessary and sufficient condition that a series of distinct natural numbers constitutes a group as regards multiplication, mod m, is that each of the numbers of the series has the same highest common factor with m and that the quotient obtained by dividing m by this highest common factor is prime to this factor.[1] The special but fundamental case when this highest common factor is unity is the one which is generally treated in the text-books.

In this special case it is very easy to see that the same group of order $\varphi(m)$ results if the operation of multiplication is replaced by that of division, when the quotient α/β, mod m, is defined as usual, as an integer γ such that $\beta\gamma \equiv \alpha$, mod m, α and β being natural numbers. The substitution on the given $\varphi(m)$ numbers which results if all of these numbers are multiplied by any one of them, for instance α, is clearly the inverse of the one which results when all of these numbers

[1] *Annals of Mathematics*, series 2, vol. 6 (1905), p. 44. It may be observed that the statement of this theorem in the *American Journal of Mathematics*, vol. 27 (1905), p. 315, is inaccurate.

are divided by α. Hence the two substitutions which result from operating with the same number correspond in one of the possible simple isomorphisms between the given groups of multiplication and division, mod m.

In the more general case noted above, when a series of distinct natural numbers constitute a group as regards multiplication, we cannot always pass directly to a group of division without further restrictions. In fact, this more general set of numbers does not include unity with respect to the modulus, while the division of a number by itself gives unity for a quotient. It is, however, easy to see that whenever a set of numbers constitutes a group as regards multiplication, mod n, these numbers must also constitute a group as regards division, mod n, provided the quotient is restricted to the given set of numbers mod n.

As the converse of this proposition is evidently also true it results that *a necessary and sufficient condition that a set of distinct natural numbers constitutes a group with respect to division, mod m, when all of these numbers are divided successively by each one of them and the quotients are restricted to numbers of the set, is that m has the same highest common factor with each of these numbers and that the quotient obtained by dividing m by this factor is prime to this factor.* For instance, the set of numbers; 2, 4, 8, 10, 14, 16 constitutes the cyclic group of order 6 with respect to each of the operations of multiplication and division mod 18, provided that in the latter case the quotients are restricted to this set of numbers. In general, the groups of multiplication and division which are obtained in this manner are simply isomorphic, and the substitutions which correspond to the same number in these two groups are the inverses of each other.

CALIFORNIA TEACHERS OF MATHEMATICS.

The following extracts from the report of the last annual meeting of the Mathematics Section of the California High School Teachers' Association by the chairman, Professor Henry W. Stager, of Fresno Junior College, are significant in many ways, and will be of interest to readers of the MONTHLY:

At the annual meeting in July, 1914, the Mathematics Section of the California High School Teachers' Association adopted an official reading course for the present school year. The purpose of this course is to arouse a greater interest in the subject on the part of teachers of mathematics rather than to increase their knowledge of mere mechanical methods of presentation. Every teacher is urged to undertake the careful reading of one or more of these books this year. The course is divided into three sections, graded according to difficulty. Each teacher can find some book suited to his individual needs. The University of California will grant credit for the study of certain of the books as part of the work in university extension. Detailed information may be obtained by addressing the University Extension Division, University of California, and referring to the course herewith.

Teachers are urged to ask their trustees to place the entire list in the school